Example Find $\int \frac{1}{(4x^2+24x+40)^{3/2}}\,dx = I$

$$4x^2+24x+40 =$$
$$4(x^2+6x+10)$$
$$= 4(x^2+6x+9-9+10)$$
$$= 4((x+3)^2+1)$$

$$\to I = \int \frac{1}{\left[4((x+3)^2+1)\right]^{3/2}}\,dx$$

$$4^{3/2}\cdot\left((x+3)^2+1\right)^{3/2}$$

$$= \frac{1}{8}\int \frac{dx}{\left((x+3)^2+1\right)^{3/2}} = \frac{1}{8}\int \frac{\sec^2\theta\, d\theta}{\underbrace{(\underbrace{\tan^2\theta+1}_{\sec^2\theta})^{3/2}}_{\sec^3\theta}}$$

let $x+3=\tan\theta$
$dx = \sec^2\theta\, d\theta$

$$= \frac{1}{8}\int \frac{1}{\sec\theta}\,d\theta$$
$$= \frac{1}{8}\int \cos\theta\, d\theta = \frac{1}{8}\sin\theta + C$$
$$= \boxed{\frac{1}{8}\frac{x+3}{\sqrt{(x+3)^2+1}} + C}$$

$\sin^2 + \cos^2 = 1$

div by $\cos^2$

$\tan^2 + 1 = \sec^2$

div by $\sin^2$

$1 + \cot^2 = \csc^2$

$\tan\theta = \frac{x+3}{1}$

$\sin\theta = \frac{x+3}{\sqrt{(x+3)^2+1}}$

Long division & Partial Fractions

Example: Find $\int \frac{x^3-4x^2+2}{4x^2+1}\,dx$

Note deg numerator of $\geq$ deg of denom.

Let's do long division:

$$\frac{x^3-4x^2+2}{4x^2+1} = (\text{quotient}) + \frac{\text{remainder}}{4x^2+1}$$

Practice: $\frac{73}{21} = (\text{quotient}) + \frac{\text{remainder}}{21} = \boxed{3+\frac{10}{21}}$

$$\begin{array}{r|l} & \;\;3.\ R10 \\ \hline 21 & 73.000 \\ & -63 \\ \hline & \;\;10. \end{array}$$

$$\begin{array}{r|l} & \frac{1}{4}x + -1 \quad \text{quotient} \\ \hline 4x^2+0x+1 & x^3-4x^2+0x+2 \\ & -(x^3+0x^2+\frac{1}{4}x) \\ \hline & \quad -4x^2 - \frac{1}{4}x + 2 \\ & \quad -(-4x^2+0x-1) \\ \hline & \qquad -\frac{1}{4}x+3 \quad \text{Remainder} \end{array}$$

$$\therefore \frac{x^3-4x^2+2}{4x^2+1} = \frac{1}{4}x - 1 + \frac{-\frac{1}{4}x+3}{4x^2+1}$$

$$= \frac{1}{4}x - 1 + \frac{-\frac{1}{4}x}{4x^2+1} + \frac{3}{4x^2+1}$$

$$\Longrightarrow \int \frac{x^3-4x^2+2}{4x^2+1}\,dx = \int\left(\frac{1}{4}x - 1 + \frac{-\frac{1}{4}x}{4x^2+1} + \frac{3}{4x^2+1}\right)dx$$

$$= \frac{1}{8}x^2 - x + \int \frac{-\frac{1}{4}x}{4x^2+1}\,dx + \int \frac{3}{4x^2+1}\,dx$$

$u = 4x^2+1$

$du = 8x\,dx$

$\frac{1}{8}du = x\,dx$

$u = 2x$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= \frac{1}{8}x^2 - x - \frac{1}{32}\int \frac{du}{u} + \frac{3}{2}\int \frac{du}{u^2+1}$$

$$= \frac{1}{8}x^2 - x - \frac{1}{32}\ln|\underset{\substack{\uparrow \\ 4x^2+1}}{u}| + \frac{3}{2}\arctan(\underset{\substack{\uparrow \\ 2x}}{u}) + C$$

$$= \boxed{\frac{1}{8}x^2 - x - \frac{1}{32}\ln(4x^2+1) + \frac{3}{2}\arctan(2x) + C}$$

Partial fractions – rewriting fractions of polynomials where the denominator can factor.

Example

$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x^2+4}$$

$$= \frac{(x+2)(x^2+4) - (x+1)(x^2+4) + (x+1)(x+2)}{(x+1)(x+2)(x^2+4)}$$

common denom

$$= \frac{\cancel{x^3} + 2x^2 + 4x + 8 - \cancel{x^3} - x^2 - 4x - 4 + x^2 + 3x + 2}{(x+1)(x+2)(x^2+4)}$$

Common denom simplifying

$$= \frac{2x^2+3x+6}{(x+1)(x+2)(x^2+4)}$$

partial fractions

Note (deg top) < (deg bottom)

How would we do partial fractions if we did not know the answer?

We know the form of the answer:

deg 2 in bottom

$$\frac{2x^2+3x+6}{(x+1)(x+2)(x^2+4)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

Using this equation → solve for A, B, C, D.

To solve: multiply by common denom

$$2x^2+3x+6 = A(x+2)(x^2+4) + B(x+1)(x^2+4) + (Cx+D)(x+1)(x+2)$$

At this point: need equations for A, B, C, D

$\boxed{\text{coefficient of } x^3 \text{ term}}$ $0 = A + B + C$

Next: plug in different #s for x to get more equations.

Or • Multiply RHS out completely, and The x^2, x, constant terms must match on both sides.